Problems on The Infinite Series

1 Problems on convergence

1. Give the ϵ-definition of convergence of an infinite series $\sum_{n\in I} a_n$, where I is an infinite set.
2. If $\sum_{n\in I} a_n$ converges then prove that the sum is unique.
3. Prove that a necessary condition for the convergence of $\sum_{n\in I} a_n$ is $\lim_{n\to\infty} a_n = 0$. Is that condition sufficient? Justify.
4. State and prove the necessary and sufficient condition for the convergence of an infinite series $\sum_{n\in I} a_n$.
5. Show that if $\sum_{n\in I} a_n$ converges, then there is a positive number M so that all the sums $|\sum_{n\in I} a_n| \leq M$ for any finite subset $I_0 \subset I$.
6. Check the convergence of the following infinite series and find their sum, if converge

$$\text{(i)} \sum_{n=1}^{\infty} \frac{1}{2^n} \quad \text{(ii)} \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}} \quad \text{(iii)} \sum_{n=1}^{\infty} \frac{1}{4n^2-1} \quad \text{(iv)} \sum_{n=1}^{\infty} sin\left(\frac{n!\pi}{720}\right) \quad \text{(v)} \sum_{n=1}^{\infty} tan^{-1}\left(\frac{1}{n^2+n+1}\right)$$

$$\text{(vi)} \sum_{n=1}^{\infty} \frac{1}{(n!)^2} \quad \text{(vii)} \sum_{n=1}^{\infty} \frac{n^2-n+1}{n!} \quad \text{(viii)} \sum_{n=1}^{\infty} \frac{n^2}{n!} \quad \text{(ix)} \sum_{n=1}^{\infty} \frac{1}{4n(n+1)(2n+1)} \quad \text{(x)} \sum_{n=1}^{\infty} \left(\frac{\sum_{k=1}^{n} k}{n!}\right)$$

7. Prove that any rearrangement of terms of an infinite positive series does not change its sum.

2 Problems on the series of non-negative terms

1. (*Comparison Test*) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of positive terms. Check the convergence of $\sum_{n=1}^{\infty} a_n$ depending upon $\sum_{n=1}^{\infty} b_n$,
   (i) If there is a natural number N such that $a_n \leq kb_n$ for all $n \geq N$, and k is a fixed positive number.
   (ii) If there is a natural number N such that $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for all $n \geq N$.
   (iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = l$, where l is a non-zero finite number. Also discuss the case when $l = 0$.
2. Determine for which value of α the following series are convergent
   $$\text{(i)} \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \quad \text{(ii)} \sum_{n=1}^{\infty} (\sqrt[n]{n}-1)^{\alpha} \quad \text{(iii)} \sum_{n=1}^{\infty} \left(1 - n\,sin\frac{1}{n}\right)^{\alpha} \quad \text{(iv)} \sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!}\right)^{\alpha}$$
3. Prove that if a series $\sum_{n=1}^{\infty} a_n$ with positive terms converges, then $\sum_{n=1}^{\infty} (p^{a_n}-1)$, where $p > 1$ also converges.
4. Suppose a series $\sum_{n=1}^{\infty} a_n$ of non-negative terms converges. Prove that the series $\sum_{n=1}^{\infty} \sqrt{a_n a_{n-1}}$ also converges but not conversely. What can you say about the converse statement when $\{a_n\}$ is monotone decreasing.
5. Assume that a positive term sequence $\sum_{n=1}^{\infty} a_n$ diverges and $\{S_n\}$ be its n^{th} partial sequence. Study the behaviour of
   $$\text{(i)} \sum_{n=1}^{\infty} \frac{a_n}{1+a_n} \quad \text{(ii)} \sum_{n=1}^{\infty} \frac{a_n}{1+n\,a_n} \quad \text{(iii)} \sum_{n=1}^{\infty} \frac{a_n}{1+n^2\,a_n} \quad \text{(iv)} \sum_{n=1}^{\infty} \frac{a_n}{1+a_n^2} \quad \text{(v)} \sum_{n=1}^{\infty} n\,a_n\,sin\left(\frac{1}{n}\right)$$
   (vi) $\sum_{n=1}^{\infty} \frac{a_n}{S_n^{\alpha}}$, where $\alpha > 0$ (vii) $\sum_{n=1}^{\infty} \frac{a_n}{S_n S_{n-1}^{\beta}}$, where $\beta > 0$
6. Let $\{a_n\}$ be a sequence of positive terms, diverging to infinity. What can you say about the convergence of the following series?
   $$\text{(i)} \sum_{n=1}^{\infty} \frac{1}{(a_n)^n} \quad \text{(ii)} \sum_{n=1}^{\infty} \frac{1}{(a_n)^{log\,n}} \quad \text{(iii)} \sum_{n=1}^{\infty} \frac{1}{(a_n)^{log\,log\,n}}$$

7. ($Cauchy's\ Condensation\ Test$) Let $\{a_n\}$ be a monotone decreasing sequence of non-negative terms. Prove that the series $\sum_{n=1}^{\infty} a_n$ converges (diverges) iff $\sum_{n=1}^{\infty} p^n a_{p^n}$ converges (diverges).
8. Test the convergence of the following series
(i) $\sum_{n=2}^{\infty} \frac{1}{n\,(log\,n)^{\alpha}}$, where $\alpha > 0$ (ii) $\sum_{n=3}^{\infty} \frac{1}{n\,(log\,n)\,(log(log\,n))}$ (iii) $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$
9. ($Logarithmic\ Test$) Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms and $\lim_{n\to\infty} n \log \frac{a_n}{a_{n+1}} = l$. Prove that $\sum_{n=1}^{\infty} a_n$ converges if $l > 1$ and diverges if $l < 1$. Also discuss the case when $l = 1$.
10. ($Raabe's\ Test$) Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms and $\lim_{n\to\infty} n \left(\frac{a_n}{a_{n+1}} - 1\right) = l$. Prove that $\sum_{n=1}^{\infty} a_n$ converges if $l > 1$ and diverges if $l < 1$. Also discuss the case when $l = 1$.
11. Study the behaviour of the following series
(i) $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}$ (ii) $\sum_{n=1}^{\infty} \frac{1}{a^{log\,n}}$, $a > 0$ (iii) $\sum_{n=1}^{\infty} \frac{1}{a^{log(log\,n)}}$, $a > 0$ (iv) $\sum_{n=1}^{\infty} a^{1+\frac{1}{2}+\ldots+\frac{1}{n}}$, $a > 0$ (v) $\sum_{n=0}^{\infty} \frac{n^n}{e^n\,n!}$
12. If $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms, then what can you say about the convergence of the series $\sum_{n=1}^{\infty} \frac{a_1 + a_2 + \ldots + a_n}{n}$?
13. ($Integral\ Test$) Let f be a non-negative decreasing function on $[1, \infty)$ such that the integral $\int_1^X f(x)\,dx$ can be computed for all $X > 1$. If $\lim_{X\to\infty} \int_1^X f(x)\,dx < \infty$ exists then the series $\sum_{n=1}^{\infty} f(n)$ converges and if If $\lim_{X\to\infty} \int_1^X f(x)\,dx = \infty$ then the series $\sum_{n=1}^{\infty} f(n)$ diverges.
14. Let f be a positive and differentiable function on $(0, \infty)$ such that f' decreases to zero. Show that the series $\sum_{n=1}^{\infty} f'(n)$ and $\sum_{n=1}^{\infty} \frac{f'(n)}{f(n)}$ either both converge or diverge.
15. ($Kummer's\ Test$) Let $\{a_n\}$ be a sequence of positive terms. Prove that
(i) If there is a sequence $\{b_n\}$ of positive numbers and a positive constant c such that $b_n \frac{a_n}{a_{n+1}} - b_{n+1} \geq c$ for all $n \in \mathbb{N}$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
(ii) If there is a sequence $\{b_n\}$ of positive numbers such that $\sum_{n=1}^{\infty} \frac{1}{b_n}$ diverges and $b_n \frac{a_n}{a_{n+1}} - b_{n+1} \leq 0$ for all $n \in \mathbb{N}$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
What can you say about the converse of this test?
16. Let $\sum_{n=1}^{\infty} a_n$ be the series of positive terms and $\frac{a_n}{a_{n+1}} = 1 + \frac{\alpha}{n} + \frac{\phi(n)}{n^{\lambda}}$, where $\lambda > 1$ and $\{\phi_n\}$ is a bounded sequence, then prove that the series $\sum_{n=1}^{\infty} a_n$ converges if $\alpha > 1$ and diverges if $\alpha \leq 1$.
17. Discuss the convergence of the hypergeometric series $\sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!} \frac{\beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)} x^n$ where α, β γ are positive constants and $x > 0$.

3 Problems on alternating series

1. ($Leibnitz's\ Test$) If $\{a_n\}$ be a monotone decreasing sequence of positive real numbers and $\lim_{n\to\infty} a_n = 0$, then prove that the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges.
2. ($Dirichlet's\ Test$) If $\{a_n\}$ be a monotone sequence converging to 0 and the n^{th} partial sum of the series

$\sum_{n=1}^{\infty} b_n$ is bounded, then prove that the series $\sum_{n=1}^{\infty} a_n b_n$ converges. Also check the convergence of $\sum_{n=1}^{\infty} a_n^k b_n$, for all natural number k if an extra condition that the series $\sum_{n=1}^{\infty} |a_{n+1} - a_n|$ is convergent be added.

3. (*Abel's Test*) Let $\{a_n\}$ be a monotone convergent sequence and the series $\sum_{n=1}^{\infty} b_n$ is convergent. prove that the series $\sum_{n=1}^{\infty} a_n b_n$ converges.
4. (*D'Alembert's Ratio Test*) Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$. Prove that $\sum_{n=1}^{\infty} a_n$ converges if $l < 1$ and diverges if $l > 1$. Also discuss the case when $l = 1$. Generalise this for the series of arbitrary terms.
5. (*Cauchy's Root Test*) Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms and $\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = l$. Prove that $\sum_{n=1}^{\infty} a_n$ converges if $l < 1$ and diverges if $l > 1$. Also discuss the case when $l = 1$. Generalise this for the series of arbitrary terms.
6. What do you mean by an absolutely convergent and an unconditionally convergent series? Prove that every absolutely convergent series is unconditionally convergent. Is the converse true? Justify.
7. What do you mean by a conditionally convergent series? Prove that every non-absolutely convergent series is conditionally convergent. Is the converse true? Justify.
8. Does the condition $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ imply that the convergence of $\sum_{n=1}^{\infty} a_n$ is equivalent of the convergence of the series $\sum_{n=1}^{\infty} b_n$?
9. Assume that a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and define $p_n = \frac{|a_n|+a_n}{2}$ and $q_n = \frac{|a_n|-a_n}{2}$ for all $n \in \mathbb{N}$. Show that both the series $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ is convergent. What can you say about $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$, if $\sum_{n=1}^{\infty} a_n$ is conditionally convergent?
10. Let $\{a_n\}$ be a sequence of positive terms converging to 0. Prove that if the series $\sum_{n=1}^{\infty} a_n b_n$ converges then $\lim_{n\to\infty} a_n (b_1 + b_2 + ... + b_n) = 0$.
11. Let α be a given positive number. Prove that if the series $\sum_{n=1}^{\infty} \frac{a_n}{n^\alpha}$ converges then $\lim_{n\to\infty} \frac{a_1 + a_2 + ... + a_n}{n^\alpha} = 0$.
12. Does there exists a convergent series $\sum_{n=1}^{\infty} a_n$ such that all the series of the form $\sum_{n=1}^{\infty} a_n^k$, where $k \in \{2, 3, 4, ...\}$, diverge?
13. Decide whether the series are convergence, absolutely convergent, conditionally convergent, unconditionally convergent or divergent.
    (i) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$ (ii) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n(n+1)}$ (iii) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$ (iv) $\sum_{n=1}^{\infty} \frac{(-1)^{\sqrt{n}}}{n^\alpha}$, $\alpha \in \mathbb{R}$
    (v) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} sin\left(\frac{1}{n}\right)$ (vi) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} cos\,(n\alpha)$, $\alpha \in \mathbb{R}$ (vii) $\sum_{n=1}^{\infty} (-1)^n \frac{n\,log\,n}{e^n}$ (viii) $\sum_{n=1}^{\infty} sin\left(\frac{\pi}{n}\right)$
    (ix) $\sum_{n=1}^{\infty} \frac{(-1)^{\{\sqrt{n}\}}(n+1)^n}{n^{n+\frac{3}{2}}}$ (x) $\sum_{n=1}^{\infty} \left(\sqrt[3]{n^3+1} - n\right)$ (xi) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n+1} - \sqrt{n}}{n}$ (xii) $\sum_{n=1}^{\infty} \frac{(-1)^{[\sqrt{n}]} n^n}{(n+1)^{n+1}}$
    (xiii) $\sum_{n=1}^{\infty} \frac{sin\left(n^{\frac{3}{2}}\right)}{n^{\frac{3}{2}}}$ (xiv) $\sum_{n=1}^{\infty} \frac{\frac{1}{2} + (-1)^n}{n}$ (xv) $\sum_{n=1}^{\infty} n^3 \left(\int_{-\pi}^{\pi} f(t)\, sin\, nt\, dt\right)$, where $f \in c^1[-\pi, \pi]$
14. Find the sum of the series
    (i) $\left(\frac{4}{20} + \frac{4.7}{20.30} + \frac{4.7.10}{20.30.40} + ...\right)$ (ii) $\left(\frac{1}{6} + \frac{5}{6.12} + \frac{5.8}{6.12.18} + \frac{5.8.11}{6.12.18.24} + ...\right)$ (iii) $\left(\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \frac{7}{4.5.6} + ...\right)$
    (iv) $\left(\frac{1}{2.3.4} + \frac{1}{4.5.6} + \frac{1}{6.7.8} + ...\right)$ (v) $\left(\frac{1}{3!} + \frac{4}{4!} + \frac{9}{5!} + ...\right)$ (vi) $\left(\frac{1}{2.3}2 + \frac{2}{3.4}2^2 + \frac{3}{4.5}2^3 + ...\right)$
    (vii) $\left(\frac{5}{3.6}\frac{1}{4^2} + \frac{5.8}{3.6.9}\frac{1}{4^3} + \frac{5.8.11}{3.6.9.12}\frac{1}{4^4} + ...\right)$ (viii) $\left(1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + ...\right)$ (ix) $\left(1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + ...\right)$

(x) $\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...\right)$ (xi) $\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+...\right)$ (xii) $\left(\frac{1}{5}+\frac{1}{3}\frac{1}{5^3}+\frac{1}{5}\frac{1}{5^5}+...\right)$

4 Problems on Cauchy product of infinite series

1. If atleast one of the following convergent series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be absolutely convergent then prove that the series of their Cauchy product $\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)$ converges. Moreover if $\sum_{n=1}^{\infty} a_n = A$ and $\sum_{n=1}^{\infty} b_n = B$, then prove that $\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) = AB$.
2. Prove that if the Cauchy product $\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)$ of two convergent series $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$, converges to C then $C = AB$.
3. Prove that the Cauchy product of two convergent series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converges iff $\lim_{n\to\infty}\sum_{k=1}^{n} a_k\left(b_n + b_{n-1} + ... + b_{n-k+1}\right) = 0$
4. Find the sum of the series
   (i) $\sum_{n=0}^{\infty} nx^{n-1}$, $|\,x\,| < 1$ (ii) $\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} x^k y^{n-k}\right)$, $|\,x\,| < 1$, $|\,y\,| < 1$ (iii) $\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{k(k+1)(n-k+1)!}\right)$.
5. Fin the sum of the Cauchy product for the following series
   (i) $\sum_{n=0}^{\infty}\frac{2^n}{n!}$, and $\sum_{n=0}^{\infty}\frac{1}{2^n n!}$ (ii) $\sum_{n=0}^{\infty}(-1)^n\frac{1}{n}$, and $\sum_{n=0}^{\infty}\frac{1}{3^n}$ (iii) $\sum_{n=0}^{\infty}(n+1)x^n$, and $\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$
   (iv) $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(n!)^2}$, and $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(n!)^2}$
6. What do you mean by a double series and its convergence?

5 Problems on rearrangement of infinite series

1. If $\sum_{n=0}^{\infty} a_n$ be a series of arbitrary terms and $\sum_{n=0}^{\infty} b_n$ is obtained from $\sum_{n=0}^{\infty} a_n$ by grouping its terms then prove that the convergence of $\sum_{n=0}^{\infty} a_n$ implies to the convergence of $\sum_{n=0}^{\infty} b_n$ to the same sum. What can you say about the convergence of $\sum_{n=0}^{\infty} a_n$, when $\sum_{n=0}^{\infty} b_n$ is convergent?
2. Prove that the rearrangement series of an absolutely convergent series converges to the same sum.
3. Show that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...$ converges to $log\,2$. Also find the rearrangements of the above series so that the rearrangement series converge to $\frac{3}{2}log\,2$, 0 and $\frac{1}{2}log\,12$ respectively.
4. Rearrange the terms of $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^p}$, $p\in(0,1)$ to increase its sum by l.

6 Problems on infinite products

1. Find the value of
   (i) $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$ (ii) $\prod_{n=1}^{\infty}\left(\frac{e^{\frac{1}{n}}}{1+\frac{1}{n}}\right)$ (iii) $\prod_{n=2}^{\infty}\left(1+\frac{1}{n^2}+\frac{1}{n^4}+\frac{1}{n^6}+...\right)$
2. Study the convergence of
   (i) $\prod_{n=1}^{\infty} n\,sin\left(\frac{1}{n}\right)$ (ii) $\prod_{n=1}^{\infty}\sqrt[n]{n}$ (iii) $\prod_{n=1}^{\infty} n\,log\left(1+\frac{1}{n}\right)$